When is a Periodic Function the Curvature of a Closed Plane Curve?

From an article of J. Arroyo, O. J. Garay and J. J. Mencia

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When does γ_k close up ?

Problem :

Given a periodic function $k : \mathbb{R} \to \mathbb{R}$, when does the associate unit planar curve $\gamma_k : \mathbb{R} \to \mathbb{R}^2$ close up ?

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The case of $\rho_k \neq \rho$.

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Then we deduce

Characterization

$$\frac{1}{2\pi}\int_0^{\rho_k} k(s)\mathrm{d}s = \frac{m}{n} \in \mathbb{Q} - \mathbb{Z}.$$

Closedness criterion

The criterion

Let k : $\mathbb{R} \to \mathbb{R}$ be a smooth periodic function of minimum period ρ_k, and $\gamma_k(s)$ the associate curve, arc-length parametrised. Then $\gamma_k(s)$ close up in $[0, n\rho_k]$, with $n > 1$, iff there exists $m \in \mathbb{Z}$ such that

$$\frac{1}{2\pi}\int_0^{\rho_k} k(s)\mathrm{d}s = \frac{m}{n} \in \mathbb{Q} - \mathbb{Z}.$$

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Let us write $\theta(s) = \int_0^s k(t)\mathrm{d}t$.

Analytical proof

We would like to show

$$\forall s, \int_s^{s+\rho} \exp(i\theta(u))\mathrm{d}u = 0.$$

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Then, by computing ($j \in 0, \ldots, n-1$)

$$\int_{s+j\rho_k}^{s+(j+1)\rho_k} exp(i\theta(u))\mathrm{d}u = \cdots = \int_s^{s+\rho_k} exp(i\theta(u) + 2\pi i\frac{m}{n}j)\mathrm{d}u$$

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$$\int_s^{s+\rho} exp(i\theta(u))\mathrm{d}u = \left\{ \sum_{j=0}^{n-1} \exp(2\pi i \frac{m}{n} j) \right\} \int_s^{s+\rho_k} exp(i\theta(u))\mathrm{d}u$$

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If $gcd(m,n) = 1$, then it's 0.

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But we have $\beta_2(s) = A_{\theta_2} y(s) + b_2$ with $b_2 = \gamma(\rho_k)$ and $\theta_2 = \theta(\rho_k)$.
M_2 is a rotation of angle θ about a point p.

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We deduce that $\beta_3 = M_3(\gamma) = M_2(\beta_2) = M_2 \circ M_2(\gamma)$.
By induction, M_{k+1} is a rotation of angle $k\theta(\rho)$, so the curve closes up in $[0, n\rho_k]$.

Closing by adding or scaling

For every $\frac{m}{n} \in \mathbb{Q} - \mathbb{Z}, gcd(m,n) = 1$ there exist constants a_n^m and b_n^m such that

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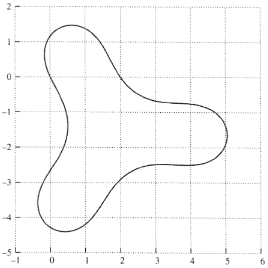
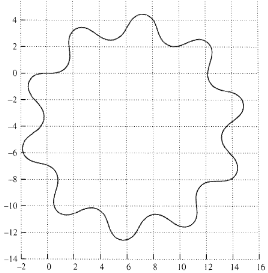
If $\theta(\rho_k)! = 0$, The plane curve with curvature $a_n^m k(s)$ close up after n periods of it's curvature with rotation index m.

Examples





Respectively $k(s) = \frac{1}{3} + sin(s)$ and $k(s) = \frac{1}{10} + sin(s)$.

Conclusion : Remaining questions

The other cases

What happens when the period of the curve and curvature are the same ?